Critical Excitation Spectrum of Quantum Chain With A Local 3-Spin Coupling

JOHN F. McCABE
2331 Gales Court, Scotch Plains, NJ 07076, USA
jfmccabe@lycos.com

TOMASZ WYDRO
Statistical Physics Group, P2M Dpt, Institut Jean Lamour
Nancy Université, Université Paul Verlaine - Metz
BP 70239, 54506 Vandoeuvre-les-Nancy Cedex, France
wydro@lpm.u-nancy.fr

July 16, 2010

Abstract

This article reports a measurement of the low-energy excitation spectrum along the critical line for a quantum spin chain having a local interaction between three Ising spins and longitudinal and transverse magnetic fields. The measured excitation spectrum agrees with that predicted by the (D_4, A_4) conformal minimal model under a nontrivial correspondence between translations at the critical line and discrete lattice translations. Under this correspondence, the measurements confirm a prediction that the critical line of this quantum spin chain and the critical point of the 2D 3-state Potts model are in the same universality class.

The solution of the two-dimensional (2D) 8-vertex model was a breakthrough in the study of lattice models in statistical mechanics [1]. Soon after the publication of this solution, the 2D 8-vertex model was shown to be equivalent to a model with local interactions between four Ising spins [2]. This equivalence stimulated studies of other models with local interactions between three and four Ising spins. One such 2D model is described by the Hamiltonian, H:

$$H = -\sum_{i,j}(J_1 \cdot S_{i,j}S_{i+1,j}S_{i+2,j} + J_2 \cdot S_{i,j}S_{i,j+1} + hS_{i,j}) \ . \tag{1}$$

In Eq. (1), J_1 and J_2 are local 3-spin couplings in the respective "x" and "y" directions, and h is a magnetic field. For $J_1 \neq J_2$, the couplings between the Ising spins are anisotropic. In the extreme anisotropic limit, a 2D model's critical behavior is also described by a quantum spin chain [3]. The quantum spin chain for the 2D model of Eq. (1) has a local interaction between three

Ising spins and interactions with a magnetic field [4]. This quantum spin chain is described by a Hamiltonian, $\widehat{H}$, given by:

$$\widehat{H} = -\sum_i \sigma_i^z \sigma_{i+1}^z \sigma_{i+2}^z - h_l \sum_i \sigma_i^z - h_t \sum_i \sigma_i^x \ . \tag{2}$$

In Eq. (2), σ_i^x, σ_i^y, and σ_i^z are the x, y, and z Pauli matrices at site i, and h_l and h_t are the respective longitudinal and transverse components of the magnetic field.

In the h_l and h_t parameter plane, the quantum spin chain of Eq. (2) has a rich phase structure for special values of the length, L, of the chain. For the special values, L = 3N with N being any positive integer [4]. For these lengths, L, the quantum spin chain has a 3-fold degenerate ground state in a region of the h_l and h_t plane. The region is bounded by a segment of the h_l-axis for which $-3 \leq h_l \leq 0$, a segment of the h_t-axis for which $0 \leq h_t \leq 1$, and a curved line segment connecting the (h_l, h_t) points (-3, 0) and (0, 1).

Along the curved line segment, there is evidence that the quantum spin chain of Eq.(2) is critical [4]. Indeed, Penson et al provided evidence identifying this critical line corresponds to a 2D conformal field theory (CFT) [5]. The CFT has a central charge, c, of 0.8, a magnetic exponent y_h of about 1.874, and a ν exponent of about 0.839. These measured values are close to the values $c = 4/5$, $y_h = 28/15$, and $\nu = 5/6$ at the critical point of the 2D 3-state Potts model. For that reason, Penson et al concluded that the critical line of the quantum spin chain of Eq. (2) is in the universality class of the critical point of the 2D 3-state Potts model.

The critical point of the 2D 3-state Potts model is described by the (D_4, A_4) conformal minimal model of the ADE classification [6]. The identification of a critical line of the quantum spin chain of Eq. (2) with the (D_4, A_4) CFT is surprising, because the (D_4, A_4) CFT has the Z_3 symmetry of the 3-state Potts model [7]. While the 3-state Potts model has a 3-state spin that causes the Z_3 symmetry, there is only a 2-state Ising spin in quantum spin chain of Eq. (2). For this reason, the quantum spin chain of Eq. (2) cannot have a Z_3 internal symmetry. Due to the absence of such an internal symmetry, the identification of the curved critical line of the quantum spin chain of Eq. (2) with the (D_4, A_4) CFT is worthy of further study.

This article evaluates the spectrum of low-lying excitations at the curved critical line of the quantum spin chain of Eq. (2) when periodic boundary conditions are imposed. In particular, both energies and momenta of excitations are determined for chains of triple integer lengths, i.e., L = 3N with N being an integer.

In this article, values of $h_l(L)$ and $h_t(L)$ are defined through the phenomenological renormalization group (PRG) [8]. When evaluated at PRG values of $h_l(L)$ and $h_t(L)$, physical quantities scale with the size of the system. The scaling behavior enables the extraction of the values of such physical properties at critical points and critical lines. For the quantum spin chain of Eq. (2), the equation for the PRG was used in the form:

$$(L-3) \cdot m(h_l(L), h_t(L), L-3) = L \cdot m(h_l(L), h_t(L), L) \ . \tag{3}$$

In Eq. (3), m(h_l, h_t, L) is the energy gap, which is defined by $m(h_l, h_t, L) = [E_1(h_l, h_t, L) - E_0(h_l, h_t, L)]$. Here, $E_0(h_l, h_t, L)$ and $E_1(h_l, h_t, L)$ are the energies of the ground state "0" and the first excited state "1" of the quantum

spin chain of length L. For our measurements, $h_l(L)$ was arbitrarily set to -1, and $h_t(L)$ was determined by solving Eq. (3). For this selection of $h_l(L)$, the PRG values of $h_t(L)$ were found to be 1.1055, 1.05425, 1.04584, and 1.04277 for L equal to 6, 9, 12, and 15, respectively. The corresponding sequence of ($h_l(L)$, $h_t(L)$) points seems to converge to a point on the curved critical line of reference [4] thereby showing that our methods reproduce properties at the curved critical line of Penson et al.

Since the Hamiltonian, $\widehat{H}$, of Eq. (2) is invariant under translations by a single lattice site, the Hamiltonian, $\widehat{H}$, was separately diagonalized over eigenspaces of the operator T_{1ls}, which generates such translations. Each eigenspace of T_{1ls}, corresponding to a momentum eigenvalue p, includes a set of one or more one-dimensional (1D) Bloch states, i.e., $\Phi_{p,i}(x)$'s. Under a translation by a single-lattice site, such a Block state satisfies: $\Phi_{p,i}(x+1) = exp[(2\pi ip/L)]\Phi_{p,i}(x)$ where $exp[(2\pi ip/L)$ is the eigenvalue of one lattice-site translation operator T_{1ls}. Here, "p" is a modular integer of (-L/2, +L/2), which defines the discrete lattice momentum, i.e., an eigenvalue, P_{1ls}, associated with T_{1ls}, and i is an integer indexing the one or more state with the same eigenvalue P_{1ls}. By using 1D Block states, both energies and the discrete lattice momenta may be measured and then, compared with the predictions of the (D_4, A_4) CFT.

At criticality, CFT predicts the scaling of excitation energies and momenta with the chain length, L. For an excitation state "i", the normalized excitation energy, E_i, will be of the form [9]:

$$E_i = [E_i(L) - E_0(L)]/E_0(L) = \frac{(\Delta_i + \bar{\Delta}_i) - (\Delta_0 + \bar{\Delta}_0)}{(\Delta_0 + \bar{\Delta}_0)} \;, \tag{4}$$

and the normalized excitation momentum, P_i, will be of the form:

$$p_i(L) = (\Delta_i - \bar{\Delta}_i) \;. \tag{5}$$

Here, Δ_i and $\bar{\Delta}_i$ are the respective left and right conformal dimensions of the excitation "i". In Eq. (4), the excitation energy, E_i, has been normalized with respect to ground state energy to remove the leading dependence on the chain length, L. In Eq. (5), the excitation momentum, P_i, has been normalized to remove the dependency on the chain length, L. These normalizations simplify comparisons between quantum spin chains of different length, L.

For the conformal minimal models, the ADE classification provides the excitation spectrum [6, 10]. For the (D_4, A_4) CFT, the ADE classification predicts that a state "i" will have a left conformal dimension Δ_i, a right conformal dimension $\bar{\Delta}_i$, a normalized excitation energy E_i, a normalized excitation momentum P_i, and a degeneracy g(i) as shown in Table 1.

Table 1: Low-lying States of the (D_4, A_4) CFT

State i	Δ_i	$\bar{\Delta}_i$	E_i	P_i	g(i)	Z_3 representation
A	0	0	0	0	1	1 singlet
B	1/15	1/15	1	0	2	1 doublet
C	2/5	2/5	6	0	1	1 singlet
D	16/15	1/15	8.5	1	2	1 doublet
E	1/15	16/15	8.5	-1	2	1 doublet
F	2/3	2/3	10	0	2	1 doublet
G	7/5	2/5	13.5	1	2	2 singlets
H	2/5	7/5	13.5	-1	2	2 singlets

By solving PRG Eq. (3), the low-lying excitation spectrum of the quantum spin chain of Eq. (2) was measured for chains having lengths, L, of 9, 12, and 15. For each value of the lattice momentum, P_{1ls}, the excitation energies were determined with a commercial linear algebra package of Maple$_{TM}$. For the PRG solution with L = 15, Figure (1) plots the measured excitation energies as a function of lattice momenta, i.e. P_{1ls}'s.

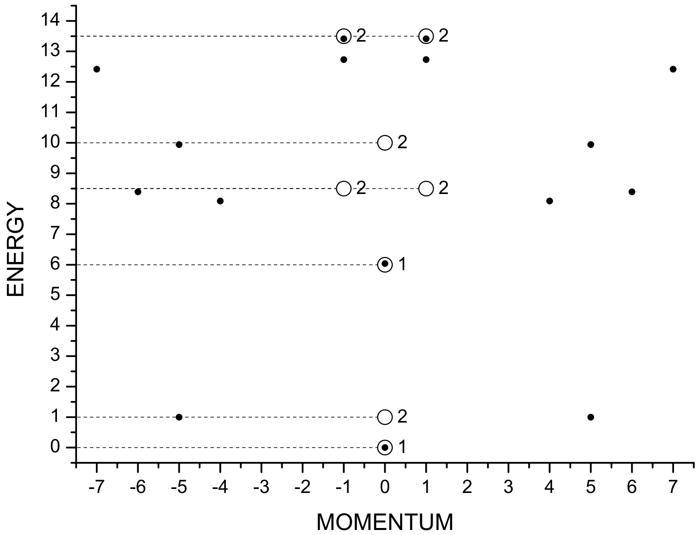

Figure 1: Measured energies and momenta, i.e. P_{1ls}'s, of low lying states of the 3-spin quantum chain of length 15 (black dots) and predicted energies and momenta of the (D_4, A_4) CFT (empty circles).

An inspection of Figure (1) shows that measured excitation energies and degeneracies agree reasonably well with predictions of the (D_4, A_4) CFT for the states A - H of Table 1. But, such an inspection also shows clear differences between the observed lattice momenta, i.e., the P_{1ls}'s, and the predicted normalized momenta as shown in Table 1. Since the predicted momenta do

not correspond to the P_{1ls}'s associated with the single lattice-site translation operator T_{1ls}, a single lattice-site translation cannot be the discrete version of a translation at the curved critical line of the quantum spin chain of Eq. (2). Based on the differences in the momentum spectra, another discrete symmetry of the quantum spin chain must provide the discrete version of a translation at the curved critical line.

The form of such translations is elucidated by Figure (2), which provides a second plot of the spectra for the PRG solution of Eq. (3) when L = 15. The second plot shows measured excitation energies as a function of 1/3 of the momentum, P_{3ls}, associated with a translation by three lattice-sites. Here, the measured momenta are $P_{3ls}/3$, which includes a division by three to produce integers as in Table 1.

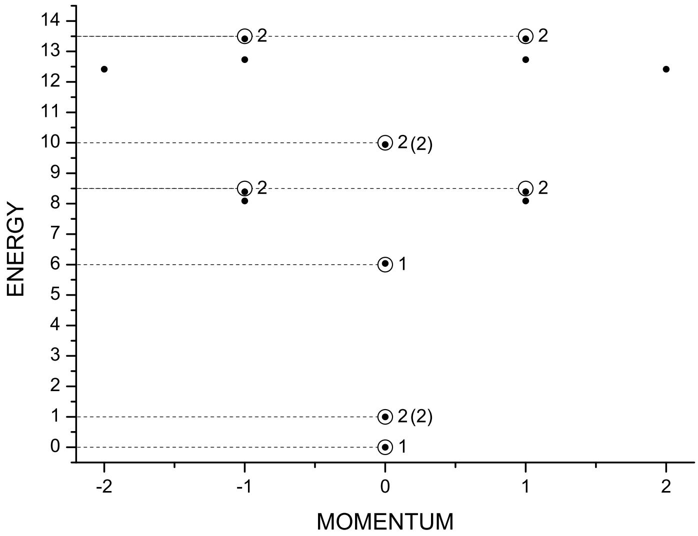

Figure 2: Measured energies and folded momenta, i.e., P_{3ls}'s, of low-lying states of the 3-spin quantum chain of length 15 (black dots) and predicted energies and momenta of the (D_4, A_4) CFT (empty circles).

For a state "i", the lattice momentum $P_{3ls}(i)$ solves:

$$exp^{(2\pi i P_{3ls}(i)/L)} = exp^{(6\pi i P_{11s}(i)/L)} \tag{6}$$

where $P_{11s}(i)$ is the momentum associated with a translation by a single lattice-site. Since values of discrete momenta are required to be in (-L/2, +L/2) for a chain of length L, Eq. (6) implies that $P_{3ls}(i) = (P_{11s}(i) + N(i))/3$ where N(i) is an integer that is fixed by requiring that P_{3ls}(i) belong to (-L/2, +L/2). The last equation produces the P_{3ls}(i) values by performing a folding operation on P_{11s}(i) values. The folding operation qualitatively changes the excitation spectrum. After the folding operation, the measured energies and momenta agree well with the predicted values of the (D_4, A_4) CFT. Thus, for the quantum

spin chain of Eq. (2), a translation by three lattice-sites is the discrete version of a translation at the curved critical line.

Figure (2) also shows that the D and E states are not exact energy doublet states. In contrast, the (D_4, A_4) CFT predicts that these states are doublets under a discrete Z_2-symmetry as shown in Table 1. Thus, the discrete Z_3-symmetry of the (D_4, A_4) CFT is broken at the PRG solutions.

For L = 9, 12, and 15, the measured excitation energies of the quantum spin chain Eq. (2) were also fitted to scaling forms to evaluate scaling behaviors as $L \to \infty$. For each state "i", the measured excitation energy, $E_i(L)$, was fit to a scaling equation $E_i(L) = E_i(\infty) + x_i/L^y$ where the last term is the leading finite-length correction. The parameters x_i and y vary with the state "i", because scaling corrections depend on the states' conformal dimensions. For the state "i", $E_i(\infty)$ is the limit of the normalized excitation energy when $L \to \infty$. The fitted values of the $E_i(\infty)$'s are shown in Table 2.

Table 2: Fitted Excitation Energies for $L \to \infty$

State	g(i)	$P_{3ls}(i)$	Fitted E_i
C	2	0	1
C	1	0	6.1
D	1	1	8.5
D	1	1	9.0
E	1	-1	8.5
E	1	-1	9.0
F	2	0	10.4
G	1	1	13.5
G	1	1	14.5
H	2	-1	13.5
H	2	-1	14.5

The fitted excitation energies of Table 2 and the predicted excitation energies of Table 1 agree well for normalized energies of about 12 or less, e.g., errors are less than about 6 percent. For higher energies, measurements are needed on quantum spin chains with longer lengths, L. For example, in Figure 2, the states with energies of about 12.3 and momenta of ± 2 were determined to converge to states with energies higher than 15 as $L \to \infty$ by comparing solutions of the PRG for L = 9, 12, and 15.

In conclusion, our measurements of the excitation energies and lattice momenta of low-lying excitations indicate that the curved critical line of the 3-spin quantum chain is in the same universality class as the critical point of the 2D 3-state Potts model, i.e., both being described by the (D_4, A_4) CFT. But, the relationship between discrete lattice translations and translations of the critical theory is nontrivial. Also, the discrete Z_3-symmetry of the conformal field theory is only recovered as the chain length becomes infinite.

References

[1] R.J. Baxter, Phys. Rev. Lett. 26 (1971) 832; ibid, Ann. Phys. (N.Y.) 70, 193 (1973).

[2] F.Y. Wu, Phys. Rev. B4 (1971) 2312; L.P. Kadanoff and F.J. Wegner, Phys. Rev. B4 (1971) 3989.

[3] E. Fradkin and L. Susskind, Phys. Rev. D 17 (1978) 2637; M. Suzuki, Prog. Theor. Phys. 46 (1971) 1337;

[4] K.A. Penson, J.M. Debierre, and L. Turban, Phys. Rev. B 37 (1988) 7884.

[5] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333; D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; ibid, Comm. Math. Phys. 107 (1986) 535.

[6] A. Cappelli, C. Itzykson, and J.-B. Zuber, Nucl. Phys. B 280 (1987) 445; ibid, Comm. Math. Phys. 113 (1987) 1; A. Kato, Mod. Phys. Lett. A 2 (1987) 585; for a review see e.g., C. Itzykson and J.-M. Drouffe, Statistical Field Theory (Cambridge University Press, UK 1989) chapter IX.

[7] J. McCabe and T. Wydro, J. Mod. Phys. A 13 (1998) 1013.

[8] M.P. Nightingale, Physica 83 A (1976) 561; B. Derrida and L. de Seze, J. Physique 43 (1982) 475; see, e.g., M.P. Nightingale, in Finite Size Scaling and Numerical Simulation of Statistical Systems (World Scientific Publishing, Singapore, Ed. V. Privman 1990) Ch. VII; ; for a brief review see, e.g., M. Henkel, Conformal Invariance and Critical Phenomena(Springer-Verlag, Germany 1999) chapter 3.

[9] J.L. Cardy, J. Phys. A 17 (1984) L385.

[10] A. Rocha-Caridi in Vertex Operators in Mathematics and Physics (Springer, New York, Eds. J. Lepowski, S. Mandelstam, and I.M. Singer 1985) 451.